UNDERSTANDING EXPONENTIAL GROWTH WITH TECHNOLOGY

Jaime Carvalho e Silva
Departamento de Matemática, Universidade de Coimbra
Apartado 3008, 3000 Coimbra, Portugal
email: jaimecs@mat.uc.pt

One of the less well understood concepts related to the exponential function is the concept of "exponential growth". A lot of people in everyday life speaks about it since Thomas Malthus (1766-1834) used it to describe population growth, but very few understand it; they want to say that the growth is very fast, but they confuse growth with the real value and ignore that there are other kinds of growth that are also very fast. It is my idea, and I experienced it with students and secondary school teachers, that technology (calculators or computers) can contribute to a better understanding of "exponential growth" and how it can be compared with other kinds of growth.

I think a student learns better a difficult concept when he studies, with some detail, significant examples or problems. Examples that have already some meaning to him, where he can feel at ease discussing with the teacher because he is aware of the context. Even if before the discussion he has wrong ideas about it or if he misses some important points, because then the teacher can attract his attention, and he will more easily teach the student a new concept. Also, when it comes to the use of technology it is very important to contact with significant examples that show how technology can be used or misused.

To illustrate these ideas I will discuss several interesting examples that cannot be fully understood without an extensive use of technology.

Example 1: Using a graphing calculator or a computer, compare the graphs of 2^x and x^2.
If we draw these two graphs by hand or with some automatic software that chooses the range for us we will normaly get something like figure 1.

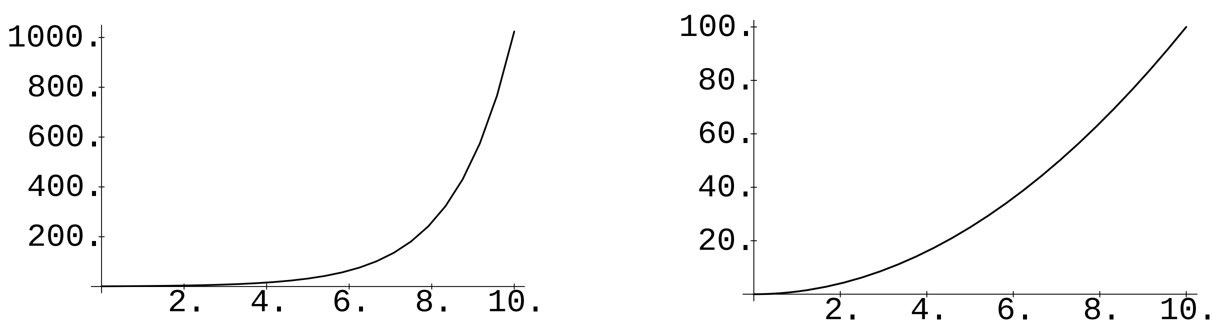

Figure 1 - The graphs of 2^x and x^2

If we draw these graphs at the same time we may get a very strange answer. If the domain is [0,3] we may be surprised to get the exponential under the power function. If we choose a large interval like [0,50] the power function disappears. Only after some experimenting we will able to obtain what is shown in figure 2.

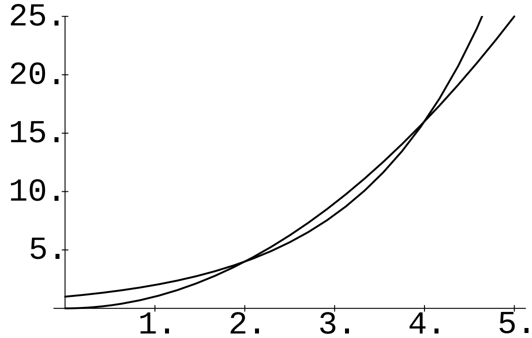

Figure 2 - The graphs of 2^x and x^2

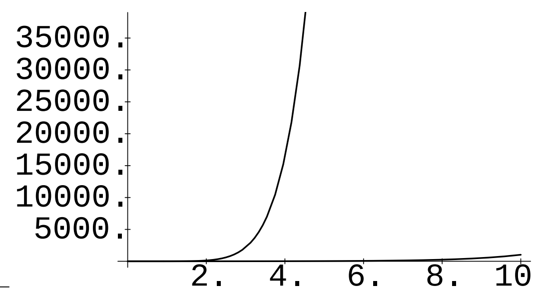

Figure 3 - The graphs of 2^x and x^7

But if we try to compare the graphs of 2^x and x^7 things get more complicated and we will surely get what is shown in figure 3 where the exponential is clearly under the power function. It will be much more difficult to get a domain and range where we will be able to see the exponential function crossing over the power function like figure 4 shows.

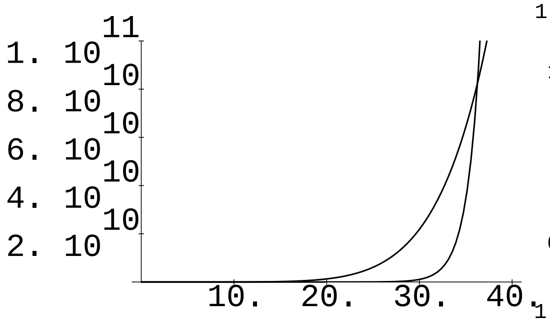

Figure 4 - The graphs of 2^x and x^7

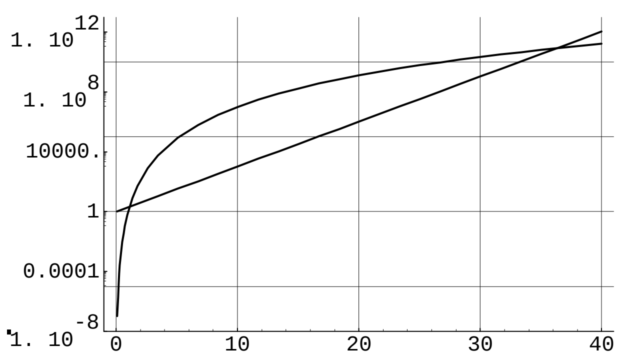

Figure 5 - The same graphs on a log scale

And it will be impossible to view the two intersecting points of the two graphs unless we use a log scale. This is surely the occasion of a very nice discussion about what can be concluded from the graphs we can draw by hand or with a graphing calculator or a computer.

But another very important point is the “growth” of each function. We can conclude from the graph that at some points the power function “grows” faster than the exponential function; this one only “grows” faster for very big and very small values of x. The same conclusion can be drawn if we study the derivative of each function. So “big” does not mean “grow faster” and “exponential growth” is only a big growth for “big” values.

$$2^x / x^{100}$$

is bigger or smaller than 1. After some experimenting we see that this fraction is bigger than 1 for very small *x*, then is smaller than 1 till we arrive near 1000.

801	$5.77797 \; 10^{-50}$
851	$1.52606 \; 10^{-37}$
901	$5.6957 \; 10^{-25}$
951	$2.89384 \; 10^{-12}$
1001	19.3918
1051	$1.66822 \; 10^{14}$
1101	$1.80009 \; 10^{27}$
1151	$2.38777 \; 10^{40}$

Figure 6 - The values of x and $2^x / x^{100}$

Then it will become again bigger than one. We see here a pattern. For *x* big enough the power function becomes smaller than the exponential; if students know limits this is a good occasion to speak about

$$\lim_{x \to +\infty} \frac{2^x}{x^{1000000000000}} = +\infty$$

and similar limits. We find here something the theory guarantees us will happen, even when it becomes too difficult to make explicit calculations. We get here a nice combination between the theory and the practical calculations.

But if we want to “see” what happens it will be very difficult to draw a decent graph even with a powerful graphing Computer Algebra System. Here the solution will be a log-graph and we will be able to see the behaviour described by figure 7.

Discussions similar to the ones we had with the previous example can be repeated or reinforced here.

Example 3: Using a graphing calculator or a computer analyze the chessboard legend.
The legend says that a bored king offered everything to a sage when the latter invented the chess game that finally was an interesting challenge to the king. But the sage, trying to prove the king was not as powerful as he himself imagined, just asked one grain of rice for the fisrt square of the chessboard, two grains for the second square, 4 grains for the third square, etc. The king promptly tried to satisfy the

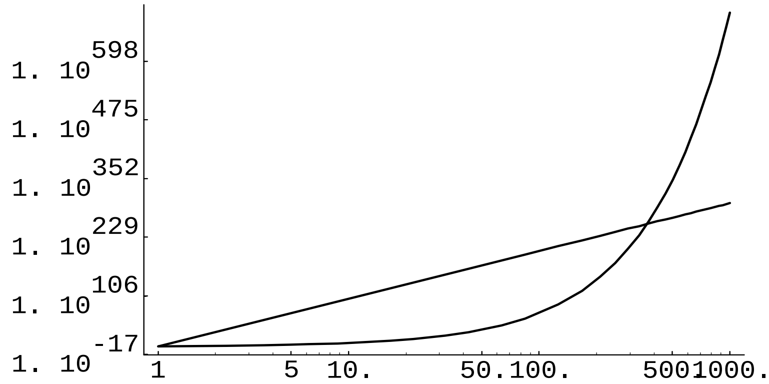

Figure 7 - The graphs of 2^x and x^{100} on a log scale

This nice story can easily be verified with the table of values of the function 2^x for x integer, or using a spreadsheet. When I was a child, this story impressed me a lot. Maybe today students are not that impressed by this old story. If so, use the next example.

Example 4: Using a graphing calculator or a computer analyze the popular way of getting a lot of money by mail (pyramid procedure).

This year I received three times a letter by email. The same letter. It told me I had to send 20 identical letters in 4 days, or else I would be very unlucky. Letters like this are very common, by email or normal mail. And it is very simple and entertaining to prove mathematically these letters are false, and to prove why some email administrators forbid the use of these letters.

We mut begin with some data. Every letter mentions at least one date. For example, the letter I mentioned had:

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>    This message has been sent to you for good luck. The original is in
> New England. It has been sent around the world nine times.
(...)
>   Do note the following: Constantine Dias received this chain in 1958.
> He asked his secretary to make twenty copies and send them out.
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So it is said that the letter began at least in 1958. Let's suppose it takes 2 weeks from the moment a letter is received by one person till it is received by the next person. Now, let's put these data on a spreadsheet for example and obtain what is shown in figure 8. We conclude that after 52 weeks there are already

6 710 886 400 000 000 000 000 000 000 000 000

people sending letters. Acording to the 1992 edition of the "World Almanac and Book of Facts" there are about

5 333 000 000

people at the surface of the earth. So, this chain of letters could not have lasted more than 16 weeks! (and we would have to include babies and illeterate people which is absurd!)

2	20
4	400
6	8000
8	160000
10	3200000
12	64000000
14	1280000000
16	25600000000
18	512000000000
20	10240000000000
22	204800000000000
24	4096000000000000
26	81920000000000000
28	1638400000000000000
30	32768000000000000000
32	655360000000000000000
34	13107200000000000000000
36	262144000000000000000000
38	5242880000000000000000000
40	104857600000000000000000000
42	2097152000000000000000000000
44	41943040000000000000000000000
46	838860800000000000000000000000
48	16777216000000000000000000000000
50	335544320000000000000000000000000
52	6710886400000000000000000000000000

Figure 8 - The number of letters sent after 52 weeks

Not even in the year 2025, when the book makes an estimate of 8 177 100 000 people it would be possible to have a six month chain. And in the end of the year 1970 we would need

559936185544451052639360570142111069530411374308662383724997275240947967795040236345219373317
901778944000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
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people. Not even in a billion years will there be enough people to send all these letters!

This problem is so interesting that we can continue the discussion: what happens if we consider people could receive the letters more than once and then had to send them again? (more or less 42 084 letters should be sent each day by every person on earth!) What happens if we consider that people who already received letters do not have to send them again? (supposing only two people had to send letters each time, then we could arrive at the end of 1958 but not even at the middle of 1959) If a student is not surprised with these conclusions, then nothing will ever surprise him! In fact this is a *mathematical modelling* problem whose key concept is *exponential growth.*